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# On inertial flow over topography. Part 2. Rotating-channel flow near the critical speed

By L. J. PRATT

Graduate School of Oceanography, University of Rhode Island, Kingston, R.I. 02882-1197

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A narrow flow passing over an obstacle in a rotating channel is analysed. When the upstream Froude number of the flow approaches unity and the obstacle height is sufficiently small, stationary Kelvin waves may appear in the channel. Under these conditions the usual nonlinear hydraulic theory (e.g. Gill 1977) must be replaced by a nonlinear dispersive theory. When the flow upstream of the obstacle is subcritical, the nonlinear dispersive theory produces three solutions, two of which resemble the solutions of hydraulic theory and a third which contains cnoidal lee waves. Upstream influence due to the obstacle becomes a function of obstacle shape as well as height. The 'controlled' solution is distinguished by the presence of a partial solitary wave in the lee of the obstacle.

## 1. Introduction

This is the second of a pair of papers dealing with topographic effects in inertial, rotating-channel flows. Applications to currents in oceanic straits have been described in the first paper (Pratt 1983*a*, hereinafter referred to as P1). As was the case in P1, all flows are confined to a thin, inviscid, homogeneous layer of fluid moving beneath a deep upper layer of slightly lower density. The upper layer is inactive and the gravitational acceleration $g'$ of the interface is reduced in proportion to the difference in densities of the two layers. The channel is assumed to rotate at constant angular speed $\Omega$ about the vertical ($z$-axis) and contain vertical walls at $y = \pm w$.

The flows considered in P1 are subject to a number of fundamental restrictions. The first is a 'long-wave' approximation governing the narrowness of the current. This approximation is formally imposed by requiring both the vertical and lateral (cross-channel) scales of motion $D$ and $(g'D)^{\frac{1}{2}}/2\Omega$ to be small compared with the along-stream scale $L$. The effect is to limit wave dispersion associated with vertical and lateral accelerations of fluid parcels. The second restriction requires all fluid parcels emanating upstream to possess uniform potential vorticity. This assumption is deemed necessary for analytic tractibility.

The assumption of vertical narrowness ($D/L \rightarrow 0$) and of uniform potential vorticity will be retained in the present treatment. Under this condition the inviscid dimensionless equations of motion governing the steady flow in the lower layer are

$$uu_x + vu_y - v = -h_x - b_x, \quad \delta^2(uv_x + vv_y) + u = -h_y - b_y, \quad (uh)_x + (vh)_y = 0, \tag{1.1$a,b,c$}$$

where the subscripts denote partial differentiation.

In addition, the conservation law for potential vorticity, derived from (1.1 $a$–$c$), is

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left[\frac{1-u_y+\delta^2 v_x}{h}\right]=0. \tag{1.1 d}$$

Here $x$ and $y$ are the along-channel and cross-channel coordinates and $u(x,y)$ and $v(x,y)$ the corresponding velocities. The thickness of the lower (active) layer is denoted by $h(x,y)$ and the elevation of the channel bottom by $b(x,y)$. The free parameter $\delta$ in (1.1 $b$) is defined as the ratio of the Rossby radius of deformation $L_d = (g'D)^{\frac{1}{2}}(2\Omega)^{-1}$ based on a depthscale $D$ and reduced gravity $g'$ to the along-channel scale $L$ of flow variation. Other scaling of variables is discussed in P1: briefly put, the Rossby number (= downstream velocity scale/$\Omega L$) is taken as unity, and the channel width $2w$ is equated with the Rossby radius $L_d$.

The boundary conditions on the rigid channel walls are

$$v(x, \pm w) = 0. \tag{1.2}$$

The vertical component of the long-wave approximation (i.e. $D/L \to 0$) is already contained in (1.1) through the hydrostatic law for pressure and the independence of the horizontal velocities $u$ and $v$ upon elevation $z$. The horizontal component of the long-wave approximation can be made by taking the limit $\delta = L_d/L \to 0$. For all oceanic currents of deformation scale width, the depthscale is much less than the width. Lateral dispersion thus has much more room to act than does vertical dispersion. This feature has been acknowledged here by taking the limit $D/L \to 0$ *before* any limit involving $\delta$ is made.

Equation (1.1 $d$) implies that the potential vorticity

$$(1-u_y+\delta^2 v_x)/h = \phi \tag{1.3}$$

is conserved along streamlines. As in P1, it will be assumed that $\phi$ is a positive constant.

In P1 it is required that $\delta = 0$, so that lateral accelerations of fluid particles become unimportant to the cross-channel momentum balance (1.1 $b$). The flows that result are dominated by a balance between nonlinear advection along the channel and gravitational effects induced by topography. The purpose of the present paper is to explore the modification of this nonlinear balance by lateral dispersive effects which are allowed to arise when the restriction $\delta = 0$ is eased. If $\delta$ is allowed to become $O(1)$, however, (1.1 $a$–$d$) become difficult to solve analytically and a numerical solution is probably necessary. Although numerical solutions will be presented in a future paper, it seems desirable to first build one's intuition by considering a problem that contains elements of nonlinear hydraulics *and* lateral wave dispersion, yet is analytically tractable. Such a problem can be constructed by considering flows for which $\delta$ is small but finite and that are disturbed by only small along-channel variations in topography. When such a flow nears the critical speed (at which Kelvin waves become stationary) a delicate balance is achieved between nonlinear effects induced by topography and lateral dispersion associated with cross-channel variations in the flow itself.

Owing to the assumed smallness of the parameter $\delta$ and its form in (1.1), the lower layer thickness, velocity and topographic elevation are written as

$$h(x,y) = h^{(0)}+\delta^2 h^{(1)}+\delta^4 h^{(2)}+\dots, \quad u(x,y) = u^{(0)}+\delta^2 u^{(1)}+\delta^4 u^{(2)}+\dots, \tag{1.4 a, b}$$

$$v(x,y) = v^{(0)}+\delta^2 v^{(1)}+\delta^4 v^{(2)}+\dots, \quad b(x) = b^{(0)}+\delta^2 b^{(1)}+\delta^2 b^{(2)}+\dots . \tag{1.4 c, d}$$

(For simplicity, the bottom topography has been allowed to vary only with $x$.)

## 2. The lowest-order dynamics: rotating hydraulics

If (1.4*a–d*) are substituted into (1.1*a*, *b*), (1.2) and (1.3), the results, to lowest order in $\delta^2$, are

$$u^{(0)}u_x^{(0)}+v^{(0)}u_y^{(0)}-v^{(0)} = -h_x^{(0)}-b_x^{(0)}, \tag{2.1a}$$

$$u^{(0)} = -h_y^{(0)}, \quad 1-u_y^{(0)} = h^{(0)} \quad v^{(0)}(x, \pm w) = 0. \tag{2.1b,c,d}$$

The horizontal velocities can be described as 'semigeostrophic', in view of (2.1*a*, *b*).

An equation for the cross-channel structure of $h^{(0)}$ can be obtained by combining (2.1*b*) and (2.1*c*). The result can be written as

$$h_{yy}^{(0)}-\phi h^{(0)} = -1. \tag{2.2}$$

The lowest-order depth and velocity are therefore given by

$$h^{(0)}(x,y) = \phi^{-1}+A^{(0)}(x)\frac{\sinh(\phi^{\frac{1}{2}}y)}{\sinh(\phi^{\frac{1}{2}}w)}+B^{(0)}(x)\frac{\cosh(\phi^{\frac{1}{2}}y)}{\cosh(\phi^{\frac{1}{2}}w)}, \tag{2.3a}$$

$$u^{(0)}(x,y) = -\phi^{\frac{1}{2}}\left[A^{(0)}(x)\frac{\cosh(\phi^{\frac{1}{2}}y)}{\sinh(\phi^{\frac{1}{2}}w)}+B^{(0)}(x)\frac{\sinh(\phi^{\frac{1}{2}}y)}{\cosh(\phi^{\frac{1}{2}}w)}\right]. \tag{2.3b}$$

Following the procedure used in P1, it is helpful to define the new independent variables†

$$\bar{h}^{(0)} = \tfrac{1}{2}[h^{(0)}(x,w)+h^{(0)}(x,-w)] = \phi^{-1}+B^{(0)}, \tag{2.4a}$$

$$\hat{h}^{(0)} = \tfrac{1}{2}[h^{(0)}(x,w)-h^{(0)}(x,-w)] = A^{(0)}, \tag{2.4b}$$

$$\bar{u}^{(0)} = \tfrac{1}{2}[u^{(0)}(x,w)+u^{(0)}(x,-w)] = -\phi^{\frac{1}{2}}T^{-1}A^{(0)}, \tag{2.4c}$$

$$\hat{u}^{(0)} = \tfrac{1}{2}[u^{(0)}(x,w)-u^{(0)}(x,-w)] = -\phi^{\frac{1}{2}}TB^{(0)}, \tag{2.4d}$$

where

$$T = \tanh(\phi^{\frac{1}{2}}w). \tag{2.5}$$

From (2.4*a–d*) it follows that

$$\bar{u}^{(0)} = -\phi^{\frac{1}{2}}T^{-1}\hat{h}^{(0)}, \quad \hat{u}^{(0)} = \phi^{\frac{1}{2}}T(\phi^{-1}-\bar{h}^{(0)}). \tag{2.6a,b}$$

The lowest-order flow rate $Q^{(0)}$ through the channel can be related to $\hat{h}^{(0)}$ and $\bar{h}^{(0)}$ by multiplying (2.1*b*) by $h^{(0)}$ and integrating across the channel. The result is

$$Q^{(0)} = \int_{-w}^{w} u^{(0)}h^{(0)}\,\mathrm{d}y = -2\bar{h}^{(0)}\hat{h}^{(0)}. \tag{2.7}$$

From this point, discussion will be restricted to flows with positive flow rates:

$$Q^{(0)} > 0. \tag{2.8}$$

Furthermore, the thickness of the lower layer is assumed to remain finite at all points within the channel, so that

$$\bar{h}^{(0)} > 0. \tag{2.9}$$

It follows from (2.7) and (2.8) that

$$\hat{h}^{(0)} < 0. \tag{2.10}$$

(It is possible for the lower layer to separate from the wall at $y = w$, in which case the layer thickness $h^{(0)}(x,w) = \bar{h}^{(0)}(x)+\hat{h}^{(0)}(x)$ vanishes. Gill (1977) has shown that the finiteness of $h^{(0)}(x,w)$,

$$\bar{h}^{(0)}(x)+\hat{h}^{(0)}(x) > 0, \tag{2.11}$$

is a necessary and sufficient condition for the finiteness of $h^{(0)}(x,y)$.)

† The notation $\Delta(\;)$ in P1 has been replaced here by $(\hat{\;})$.

The along-channel behaviour of $\bar{h}^{(0)}$, $\hat{h}^{(0)}$, $\bar{u}^{(0)}$ and $\hat{u}^{(0)}$ can be ascertained through the use of (2.1$a$) and the boundary condition (2.1$d$). The result (see Gill 1977 or P1) is the aforementioned hydraulic theory, which determines $\bar{h}^{(0)}$, $\hat{h}^{(0)}$ etc. as functions of the topographic elevation $b^{(0)}(x)$. The solutions behave in a similar way to those of classical hydraulic theory (Chow 1959). A generalized Froude number

$$F_{\mathrm{d}} = \frac{\bar{u}^{(0)}}{C^{(0)}} = -\frac{\phi^{\frac{1}{2}}T^{-1}\hat{h}^{(0)}}{\bar{h}^{(0)\frac{1}{2}}[1-T^2(1-\phi\bar{h}^{(0)})]^{\frac{1}{2}}} \tag{2.12}$$

can be defined such that $C^{(0)}$ is the speed of a small-amplitude Kelvin wave relative to the advective speed $\bar{u}^{(0)}$ of the current against which it propagates (cf. equation (3.19) of P1). The steady current is called subcritical, supercritical or critical for $F_{\mathrm{d}} < 1$, $F_{\mathrm{d}} > 1$ or $F_{\mathrm{d}} = 1$ respectively, the latter case occurring when the absolute speed $\bar{u}^{(0)} - C^{(0)}$ of the wave is zero.

At this point, one avenue of investigation would be to study the small departures of the above hydraulic solutions due to dispersion at higher orders of $\delta^2$. Aside from being extremely difficult, this approach doesn't really satisfy our desire to give dispersion an important role. Therefore variations in the lowest-order bottom topography are required to vanish, $b^{(0)} = 0$, causing $\bar{h}^{(0)}$, $\hat{h}^{(0)}$, $\bar{u}^{(0)}$ and $\hat{u}^{(0)}$ to become $x$-independent and the cross-channel velocity $v^{(0)}$ to vanish.

## 3. The $O(\delta^2)$ dynamics: linear long-wave theory

To $O(\delta^2)$ (1.1$a,b$), (1.2) and (1.3) are

$$u^{(0)}u_x^{(1)} + v^{(1)}u_y^{(0)} - v^{(1)} = -h_x^{(1)} - b_x^{(1)}, \tag{3.1a}$$

$$u^{(1)} = -h_y^{(1)}, \quad -u_y^{(1)} = \phi h^{(1)}, \quad v^{(1)}(x, \pm w) = 0. \tag{3.1b,c,d}$$

(The $x$-independence of the basic (lowest-order) fields has been considered in deriving (3.1$a$–$d$).) According to (3.1$b$), cross-stream accelerations do not come into play, and the horizontal velocities remain semigeostrophic.

Proceeding as in §2, (3.1$b$) and (3.1$c$) can be combined and the following expressions for $u^{(1)}$ and $h^{(1)}$ found:

$$h^{(1)}(x,y) = A^{(1)}(x)\frac{\sinh(\phi^{\frac{1}{2}}y)}{\sinh(\phi^{\frac{1}{2}}w)} + B^{(1)}(x)\frac{\cosh(\phi^{\frac{1}{2}}y)}{\cosh(\phi^{\frac{1}{2}}w)}, \tag{3.2a}$$

$$u^{(1)}(x,y) = -\phi^{\frac{1}{2}}\left[A^{(1)}(x)\frac{\cosh(\phi^{\frac{1}{2}}y)}{\sinh(\phi^{\frac{1}{2}}y)} + B^{(1)}(x)\frac{\sinh(\phi^{\frac{1}{2}}y)}{\cosh(\phi^{\frac{1}{2}}w)}\right]. \tag{3.2b}$$

From these it can be shown that

$$\bar{u}^{(1)}(x) = -\phi^{\frac{1}{2}}T^{-1}\hat{h}^{(1)}(x) = -\phi^{\frac{1}{2}}T^{-1}A^{(1)}(x), \tag{3.3a}$$

$$\hat{u}^{(1)}(x) = -\phi^{\frac{1}{2}}T\bar{h}^{(1)}(x) = -\phi^{\frac{1}{2}}TB^{(1)}(x). \tag{3.3b}$$

where ($\bar{\ }$) and ($\hat{\ }$) again denote sums and differences of values on the channel walls.

The along-channel structure of $\bar{u}^{(1)}$, $\hat{u}^{(1)}$, $\bar{h}^{(1)}$ and $\hat{h}^{(1)}$ can be determined using the momentum equation (3.1$a$). If this equation is evaluated at each wall, where $v^{(1)} = 0$, and the two results summed, the following equation is obtained:

$$\bar{u}^{(0)}\frac{\mathrm{d}\bar{u}^{(1)}}{\mathrm{d}x} + \hat{u}^{(0)}\frac{\mathrm{d}\hat{u}^{(1)}}{\mathrm{d}x} + \frac{\mathrm{d}\bar{h}^{(1)}}{\mathrm{d}x} = -\frac{\mathrm{d}b^{(1)}}{\mathrm{d}x}. \tag{3.4a}$$

Similarly, the difference of the momentum equations along each wall is

$$\bar{u}^{(0)}\frac{\mathrm{d}\hat{u}^{(1)}}{\mathrm{d}x}+\hat{u}^{(0)}\frac{\mathrm{d}\bar{u}^{(1)}}{\mathrm{d}x}+\frac{\mathrm{d}\hat{h}^{(1)}}{\mathrm{d}x}=0. \tag{3.4b}$$

If (2.6) and (3.3) are now used to eliminate $\bar{u}^{(0)}$, $\hat{u}^{(0)}$, $\bar{u}^{(1)}$ and $\hat{u}^{(1)}$ in (3.4), the following pair of equations for the unknowns $\bar{h}^{(1)}$ and $\hat{h}^{(1)}$ results:

$$\hat{h}^{(0)}\frac{\mathrm{d}\hat{h}^{(1)}}{\mathrm{d}x}+T^2\phi^{-1}[1-T^2(1-\phi h^{(0)})]\frac{\mathrm{d}\bar{h}^{(1)}}{\mathrm{d}x}=-T^2\phi\frac{\mathrm{d}b^{(1)}}{\mathrm{d}x}, \tag{3.5a}$$

$$\bar{h}^{(0)}\frac{\mathrm{d}\hat{h}^{(1)}}{\mathrm{d}x}+\hat{h}^{(0)}\frac{\mathrm{d}\bar{h}^{(1)}}{\mathrm{d}x}=0. \tag{3.5b}$$

Eliminating $\bar{h}^{(1)}$ from the above expressions, one finds

$$\frac{\mathrm{d}\hat{h}^{(1)}}{\mathrm{d}x}=-\frac{\hat{h}^{(0)}T^2\phi\,\mathrm{d}b^{(1)}/\mathrm{d}x}{\hat{h}^{(0)2}-\phi^{-1}T^2\bar{h}^{(0)}[1-T^2(1-\phi\bar{h}^{(0)})]} \tag{3.6}$$

For nonzero values of denominator in (3.6), $\hat{h}^{(1)}$ varies linearly with the topographic elevation $b^{(1)}$. The solution is of long-wave nature and is identical in form with the solution that would be obtained by linearizing hydraulic theory for infinitesimal topographic variations. Of more interest is the case when the denominator vanishes:

$$\phi T^{-2}\hat{h}^{(0)2}=\bar{h}^{(0)}[1-T^2(1-\bar{h}^{(0)})], \tag{3.7}$$

which, according to (2.12), is the critical condition for a Kelvin wave propagating against the basic flow. There are now two possibilities. The first occurs when the bottom elevation $b^{(1)}$ is non-constant. In this case $\mathrm{d}h^{(1)}/\mathrm{d}x$ becomes unbounded and the expansion scheme (1.4) is invalidated. This situation can be remedied, however, by expanding the thickness and velocity fields in powers of $\delta$, rather than $\delta^2$, but leaving the topographic expansion in powers of $\delta^2$ intact. The new expansion is based on the expectation that the $O(\delta^2)$ topography will produce $O(\delta)$ perturbations to the basic flow. This expectation is correct, and a nonlinear equation governing the $O(\delta)$ fields can be found. However, this equation is essentially no different from the equation of hydraulics; the $O(\delta)$ fields have long-wave character and are therefore of no further interest here. The other possibility is that no topographic variations exist ($\mathrm{d}b^{(1)}/\mathrm{d}x\equiv 0$) so that $\mathrm{d}\hat{h}^{(1)}/\mathrm{d}x$ is indeterminate in (3.6). In this case the $O(\delta^2)$ dynamics are degenerate and the only information given by (3.5) is the following relationship between $\hat{h}^{(1)}$ and $\bar{h}^{(1)}$:

$$\bar{h}^{(0)}\hat{h}^{(1)}=-\hat{h}^{(0)}\bar{h}^{(1)}+Q^{(1)}\hat{h}^{(0)}, \tag{3.8}$$

obtained by integrating (3.5*b*). Here $Q^{(1)}$ is an integration constant determining a small correction to the basic flow rate.

It is now formally assumed that the basic flow is critical and that $b^{(1)}(x)=0$. The $O(\delta^2)$ perturbations are due to stationary Kelvin waves of yet-unknown along-channel structure. To determine this structure it is necessary to proceed to the next order.

## 4. The $O(\delta^4)$ dynamics

To $O(\delta^4)$ (1.1*a*, *b*), (1.2) and (1.3) are

$$u^{(0)}u_x^{(2)}+v^{(2)}u_y^{(0)}-v^{(2)}+h_x^{(2)}=-b_x^{(2)}-u^{(1)}u_x^{(1)}-v^{(1)}u_y^{(1)}, \tag{4.1a}$$

$$u^{(0)}v_x^{(1)}+u^{(2)}=-h_y^{(2)},\quad v_x^{(1)}-u_y^{(2)}=\phi h^{(2)},\quad v^{(2)}(x,\pm w)=0. \tag{4.1b,c,d}$$

Note that the along-channel velocity $u^{(2)}$ is no longer geostrophic.

Combining (4.1 *b*) and (4.1 *c*) gives the following equation for $h^{(2)}$:

$$h^{(2)}_{yy} - \phi h^{(2)} = -(u^{(0)} v^{(1)}_x)_y - v^{(1)}_x, \tag{4.2}$$

where $v^{(1)}$ can be expressed in terms of $u^{(1)}$ and $h^{(1)}$ (using (2.1 *c*) and (3.1 *a*)) as

$$v^{(1)} = (\phi h^{(0)^{-1}})\,[u^{(0)} u^{(1)}_x + h^{(1)}_x]. \tag{4.3}$$

If (3.2), (3.3) and (3.8) are used to substitute for $h^{(1)}$ and $u^{(1)}$ in (4.3), the cross-channel velocity can be written as

$$v^{(1)} = \left\{\phi^{\frac{1}{2}} T^{-1}\left[(\bar{u}^{(0)} - u^{(0)}(y))\frac{\cosh(\phi^{\frac{1}{2}} y)}{\cosh(\phi^{\frac{1}{2}} w)} + \hat{u}^{(0)}\frac{\sinh(\phi^{\frac{1}{2}} y)}{\sinh(\phi^{\frac{1}{2}} w)}\right]\right.$$
$$\left. - \phi^{\frac{1}{2}} T \bar{h}^{(0)} \hat{h}^{(0)^{-1}}\left[(u^{(0)}(y) - \bar{u}^{(0)})\frac{\sinh(\phi^{\frac{1}{2}} y)}{\sinh(\phi^{\frac{1}{2}} w)} - \hat{u}^{(0)}\frac{\cosh(\phi^{\frac{1}{2}} y)}{\cosh(\phi^{\frac{1}{2}} w)}\right]\right\}\frac{d\hat{h}^{(1)}}{dx}. \tag{4.4}$$

At the channel walls $u^{(0)}(\pm w) = \bar{u}^{(0)} \pm \hat{u}^{(0)}$, so that the bracketed terms in (4.4) vanish at $y = \pm w$. The boundary conditions $v^{(1)}(x, w) = v^{(1)}(x, -w) = 0$ are therefore satisfied. Also note that the finiteness of $v^{(1)}$ is based entirely on the $y$-structure of the basic flow. If $u^{(0)}$ becomes $y$-independent then $\bar{u}^{(0)} = u^{(0)}$, $\hat{u}^{(0)} = 0$, and $v^{(1)}$ vanishes in (4.4).

Since the expression in braces in (4.4) is solely a function of $y$, the cross-channel velocity may be written as

$$v^{(1)} = V^{(1)}(y)\frac{d\hat{h}^{(1)}}{dx}. \tag{4.5}$$

Equation (4.2) can now be written as

$$h^{(2)}_{yy} - \phi h^{(2)} = -\left(V^{(1)} + u^{(0)}\frac{dV^{(1)}}{dy} + V^{(1)}\frac{du^{(0)}}{dy}\right)\frac{d^2\hat{h}^{(1)}}{dx^2}, \tag{4.6}$$

When the expressions for $V^{(1)}$ and $u^{(0)}$ are substituted into (4.6), the coefficient of $d^2\hat{h}^{(1)}/dx^2$ becomes a complicated combination of ratios of hyperbolic functions. Determination of a closed-form particular solution is difficult. Instead, we simply use the fact that the right-hand side of (4.6) is a separable function of $y$ and $x$, and write the particular solution symbolically as $H^{(2)}(y)\, d^2\hat{h}^{(1)}/dx^2$. Thus

$$h^{(2)}(x,y) = A^{(2)}\frac{\sinh(\phi^{\frac{1}{2}} y)}{\sinh(\phi^{\frac{1}{2}} w)} + B^{(2)}\frac{\cosh(\phi^{\frac{1}{2}} y)}{\cosh(\phi^{\frac{1}{2}} w)} + H^{(2)}(y)\frac{d^2\hat{h}^{(1)}}{dx^2}. \tag{4.7a}$$

From (4.1 *b*) the corresponding velocity is

$$u^{(2)}(x,y) = -\phi^{\frac{1}{2}}\left[A^{(2)}\frac{\cosh(\phi^{\frac{1}{2}} y)}{\sinh(\phi^{\frac{1}{2}} w)} + B^{(2)}\frac{\sinh(\phi^{\frac{1}{2}} y)}{\cosh(\phi^{\frac{1}{2}} w)}\right] + U^{(2)}(y)\frac{d^2\hat{h}^{(1)}}{dx^2}, \tag{4.7b}$$

where $U^{(2)}(y) = -dH^{(2)}/dy - u^{(0)}\,V^{(1)}$. The functions $H^{(2)}$ and $U^{(2)}$ depend on the $O(0)$ flow and will not be evaluated explicitly.

From (4.7 *a, b*) it follows that

$$\bar{h}^{(2)} = B^{(2)} + \bar{H}^{(2)}\frac{d^2\hat{h}^{(1)}}{dx^2}, \quad \hat{h}^{(2)} = A^{(2)} + \hat{H}^{(2)}\frac{d^2\hat{h}^{(1)}}{dx^2}, \tag{4.8a, b}$$

$$\bar{u}^{(2)} = -\phi^{\frac{1}{2}} T^{-1} A^{(2)} + \bar{U}^{(2)}\frac{d^2\hat{h}^{(1)}}{dx^2}, \quad \hat{u}^{(2)} = -\phi^{\frac{1}{2}} T B^{(2)} + \hat{U}^{(2)}\frac{d^2\hat{h}^{(1)}}{dx^2}, \tag{4.8c, d}$$

where $(\hat{\ })$ and $(\bar{\ })$ take on the usual meanings. From these, it follows that

$$\bar{u}^{(2)} = -\phi^{\frac{1}{2}} T^{-1} \hat{h}^{(2)} + (\bar{U}^{(2)} + \phi^{\frac{1}{2}} T^{-1} \hat{H}^{(2)}) \frac{\mathrm{d}^2 \hat{h}^{(1)}}{\mathrm{d}x^2}, \tag{4.9a}$$

$$\hat{u}^{(2)} = -\phi^{\frac{1}{2}} T \bar{h}^{(2)} + (\hat{U}^{(2)} + \phi^{\frac{1}{2}} T \bar{H}^{(2)}) \frac{\mathrm{d}^2 \hat{h}^{(1)}}{\mathrm{d}x^2}. \tag{4.9b}$$

If (4.1$a$) is now written on either wall and the results summed and differenced in the usual way, the following equations for $\hat{h}^{(2)}$ and $\bar{h}^{(2)}$ are obtained:

$$\begin{aligned} \hat{h}^{(0)} \frac{\mathrm{d}\hat{h}^{(2)}}{\mathrm{d}x} + T^2 \phi^{-1}[1 - T^2(1-\phi \bar{h}^{(0)})] \frac{\mathrm{d}\bar{h}^{(2)}}{\mathrm{d}x} &= -\phi^{-1} T \frac{\mathrm{d}b^{(2)}}{\mathrm{d}x} \\ &+ [\phi^{-\frac{1}{2}} T \hat{h}^{(0)} (\bar{U}^{(2)} + \phi^{\frac{1}{2}} T^{-1} \hat{H}^{(2)}) - \phi^{-\frac{1}{2}} T^3 (\phi^{-1} - \bar{h}^{(0)}) (\hat{U}^{(2)} + \phi^{\frac{1}{2}} T \bar{H}^{(2)})] \frac{\mathrm{d}^3 \hat{h}^{(1)}}{\mathrm{d}x^3} \\ &+ \left\{ T^4 Q^{(1)} \left( \frac{\bar{h}^{(0)}}{\hat{h}^{(0)}} \right) - \left[ 1 + T^4 \left( \frac{\bar{h}^{(0)}}{\hat{h}^{(0)}} \right)^2 \right] \hat{h}^{(1)} \right\} \frac{\mathrm{d}\hat{h}^{(1)}}{\mathrm{d}x}, \end{aligned} \tag{4.10a}$$

$$\begin{aligned} \bar{h}^{(0)} \frac{\mathrm{d}\hat{h}^{(2)}}{\mathrm{d}x} + \hat{h}^{(0)} \frac{\mathrm{d}\bar{h}^{(2)}}{\mathrm{d}x} &= [\phi^{-\frac{1}{2}} T^{-1} \hat{h}^{(0)} (\hat{U}^{(2)} + \phi^{\frac{1}{2}} T \bar{H}^{(2)}) - \phi^{-\frac{1}{2}} T (\phi^{-1} - \bar{h}^{(0)}) \\ &\times (\bar{U} + \phi^{\frac{1}{2}} T^{-1} \hat{H}^{(2)})] \frac{\mathrm{d}^3 \hat{h}^{(1)}}{\mathrm{d}x^3} - \left[ Q^{(1)} - 2 \left( \frac{\bar{h}^{(0)}}{\hat{h}^{(1)}} \right) \hat{h}^{(1)} \right] \frac{\mathrm{d}\hat{h}^{(1)}}{\mathrm{d}x}. \end{aligned} \tag{4.10b}$$

Eliminating $\hat{h}^{(2)}$ from the above expressions gives

$$\frac{\mathrm{d}\bar{h}^{(2)}}{\mathrm{d}x} = \frac{D_1 \dfrac{\mathrm{d}^3 \hat{h}^{(1)}}{\mathrm{d}x^3} + (D_2 Q^{(1)} + 2D_3 \hat{h}^{(1)}) \dfrac{\mathrm{d}\hat{h}^{(1)}}{\mathrm{d}x} + D_4 \dfrac{\mathrm{d}b^{(2)}}{\mathrm{d}x}}{\hat{h}^{(0)2} - \phi^{-1} T^2 \bar{h}^{(0)}[1 - T^2(1 - \phi \bar{h}^{(0)})]}, \tag{4.11}$$

where

$$\begin{aligned} D_1 = \phi^{-\frac{1}{2}} T^{-1} [\hat{h}^{(0)} + T^4 \bar{h}^{(0)} (\phi^{-1} - \bar{h}^{(0)})] (\hat{U}^{(2)} + \phi^{\frac{1}{2}} T \bar{H}^{(2)}) \\ - \phi^{-\frac{3}{2}} T \hat{h}^{(0)} (\phi^{-1} - \bar{h}^{(0)}) (\bar{U}^{(2)} + \phi^{\frac{1}{2}} T^{-1} \hat{H}^{(2)}), \end{aligned} \tag{4.12a}$$

$$D_2 = -(\hat{h}^{(0)} + T^4 \bar{h}^{(0)2} \hat{h}^{(0)^{-1}}), \tag{4.12b}$$

$$D_3 = \tfrac{1}{2}(3\bar{h}^{(0)} + T^4 \bar{h}^{(0)} \hat{h}^{(0)^{-2}}), \tag{4.12c}$$

$$D_4 = \phi^{-1} T^2 \bar{h}^{(0)}. \tag{4.12d}$$

Note that the coefficients $D_2$, $D_3$ and $D_4$ are required by (2.9) and (2.10) to be positive. The coefficient $D_1$ is related through $\hat{h}^{(0)}$, $\bar{h}^{(0)}$, $\hat{U}^{(2)}$, $\bar{U}^{(2)}$, $\bar{H}^{(2)}$ and $\hat{H}^{(2)}$ to the basic flow. Should the cross-channel velocity vanish over the entire channel, the functions $U^{(2)}(y)$ and $H^{(2)}(y)$ in (4.7) will also vanish, implying that $D_1 = 0$. Thus, a finite cross-channel velocity $v^{(1)}$ is required to maintain finite $D_1$. We have already seen that non-zero $v^{(1)}$ is a consequence of the $y$-structure of the basic flow.

Equation (4.11) can be compared with its $O(\delta^2)$ counterpart (3.6). In both cases the denominator of the right-hand side is zero, as required by the criticality of the basic flow. At $O(\delta^2)$, boundedness of the solution was achieved by forcing topographic variations to vanish. In equation (4.11), however, topographic variations may remain and boundedness may be achieved through the *dynamical* requirement

$$D_1 \frac{\mathrm{d}^3 \hat{h}^{(1)}}{\mathrm{d}x^3} = -(D_2 Q^{(1)} + 2D_3 \hat{h}^{(1)}) \frac{\mathrm{d}\hat{h}^{(1)}}{\mathrm{d}x} + D_4 \frac{\mathrm{d}b^{(2)}}{\mathrm{d}x}, \tag{4.13}$$

which can be simplified to

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\xi^2} = -[\pm Q^{(1)} + g(\xi)]\,g(\xi) - \tilde{b}(\xi) + \mathscr{B}^{(1)} \tag{4.14}$$

by integrating once with respect to $x$ and applying the new scaled variables

$$g = \hat{h}^{(1)}(\pm D_3/D_2), \quad \xi = x(\pm D_2/D_1)^{\frac{1}{2}}, \quad \tilde{b} = b^{(2)}(\pm D_3 D_4/D_2^2). \quad (4.15a,b,c)$$

(The + sign is chosen for $D_1 > 0$ and the − sign for $D_2 < 0$.) It will now be assumed that $D_1 > 0$, so that the + sign is appropriate. As will be shown, this assumption leads to no loss in generality.)

Since (4.13) is a momentum equation, its integral (4.14) is an energy equation with the integration constant $\mathscr{B}^{(1)}$ representing the Bernoulli constant.

## 5. Steady solutions

A homogeneous form of (4.14) has been found by Benjamin & Lighthill (1954) to describe stationary gravity waves on a shallow irrotational flow near the critical speed. Their equation can also be derived from the time-dependent Kortweg–de Vries equation (Whitham 1974) for shallow water waves propagating on a basic state of rest, the steady equation resulting from a Galilean transformation which places an observer in a frame of reference in which disturbances are stationary. The only addition to the equation of Benjamin & Lighthill is the inhomogeneous term $\tilde{b}(\xi)$ in (4.14). It is important to note, however, that (4.14) is not equivalent to the equation derived by following Kelvin waves propagating on a basic state of rest. Such a state is independent of $y$ (by (2.1$b,c$)), so that the Kelvin waves possess zero $v^{(1)}$, forcing $D_1$ to be identically zero and thereby eliminating the dispersive term from (4.13).

To study the effect of topography in (4.14) it is convenient to first rewrite the latter as the non-autonomous system

$$\frac{\mathrm{d}f}{\mathrm{d}\xi} = -[Q^{(1)} + g(\xi)]\,g(\xi) - \tilde{b}(\xi) + \mathscr{B}^{(1)}, \tag{5.1}$$

$$\frac{\mathrm{d}g}{\mathrm{d}\xi} = f(\xi). \tag{5.2}$$

Recall that $g(\xi)$ is proportional to the layer-thickness difference $\hat{h}^{(1)}$ across the channel. This difference geostrophically determines both the wall-averaged velocity $\bar{u}^{(1)}$ (through (3.3$a$)) and the wall-averaged depth $\bar{h}^{(1)}$ (through (3.8)). The new variable $f = \mathrm{d}g/\mathrm{d}\xi$ is thus proportional to the along-channel gradient of these two quantities.

Suppose first that $\tilde{b}$ is locally constant, so that the channel bottom is locally flat. Uniform flows (for which $\mathrm{d}g/\mathrm{d}\xi$ and $\mathrm{d}f/\mathrm{d}\xi$ are identically zero) occur at the stationary points of (5.1) and (5.2):

$$f_s = 0, \tag{5.3}$$

$$g_{s\pm} = -\tfrac{1}{2}Q^{(1)} \pm [(\tfrac{1}{2}Q^{(1)})^2 + \mathscr{B}^{(1)} - \tilde{b}]^{\frac{1}{2}}. \tag{5.4}$$

Two real, distinct values $g_{s+}$ and $g_{s-}$ will exist, provided that

$$\tilde{b} < (\tfrac{1}{2}Q^{(1)})^2 + \mathscr{B}^{(1)}. \tag{5.5}$$

When

$$\tilde{b} = (\tfrac{1}{2}Q^{(1)})^2 + \mathscr{B}^{(1)} \tag{5.6}$$

these values coalesce and $g_{s+}$ and $g_{s-}$ take on the single value $g_{sc}$ given by

$$g_{sc} = -\tfrac{1}{2}Q^{(1)}. \tag{5.7}$$

The uniform ($\xi$-independent) solutions described by (5.3) and (5.4) can be viewed as small corrections to the basic uniform flow. Appendix A contains an evaluation of the Froude number of the corrected flow as defined by (2.12). In particular, it is shown that the $\xi$-independent flows corresponding to $g = g_{s+}$ and $g = g_{s-}$ (as occur when (5.5) is satisfied) are respectively *slightly subcritical* and *slightly supercritical*. When (5.6) is satisfied and $g_{s+} = g_{s-}$, the corrected flow is *critical*. The uniform solutions at $O(\delta^2)$ thus arise as bifurcations from a critical state which occurs when $\tilde{b} = (\tfrac{1}{2}Q^{(1)})^2 + \mathscr{B}^{(1)}$.

Some caution should be used when applying the terms subcritical and supercritical to the corrected flow. Here we use the terms to indicate the value (relative to unity) of the Froude number that would be measured in a laboratory situation, drawing no inference concerning the ability of disturbances to propagate upstream against the flow or remain stationary. In fact, dispersion will allow for stationary waves over a continuous range of Froude numbers near unity.

The general solutions to (5.1) and (5.2) can be represented as phase-plane trajectories, for which $f(\xi)$ is plotted as a function of $g(\xi)$, with $\xi$ a parameter determining position along a given trajectory. Since the topography $\tilde{b}$ is itself a function of $\xi$, the trajectories change shape as the topography varies. Consider first the special case of a flat bottom with zero elevation. $\tilde{b}(\xi) = 0$, as shown in figure 1 ($a$). (The special parameter settings $\mathscr{B}^{(1)} = 0$, $Q^{(1)} = 1$ have been used.) By (5.3) and (5.4), uniform flows exist at $(g,f) = (0,0)$ and $(-1,0)$. In the first case the flow is slightly subcritical. This solution is represented in figure 1 ($a$) by a stable *centre* point surrounded by closed trajectories. The closed (periodic) solutions consist of cnoidal Kelvin waves with dispersion properties similar to the cnoidal waves of free-surface flow (Whitham 1974). In the small-amplitude limit these waves have length $\lambda = 2^{\frac{1}{2}}\pi[(\tfrac{1}{2}Q^{(1)})^2 + \mathscr{B}^{(1)} - \tilde{b})^{-\frac{1}{4}}$, as shown in Appendix B. In the second case the uniform flow is slightly supercritical and is represented by a saddle point bordered by solutions which grow without bound. The closed trajectory that passes through the supercritical flow represents a solitary Kelvin wave, the limit of a cnoidal Kelvin wave as the wavelength goes to infinity.

The effect of topography on the above solutions can be seen in figure 1 ($b$), where the phase plane has been plotted for $\tilde{b} = 0.16$. The trajectories represent solutions valid over a flat bottom of slightly higher elevation than that of figure 1 ($a$). They also give the *instantaneous* trajectories at position $\xi_0$ for a solution over an uneven bottom with elevation $\tilde{b}(\xi_0) = 0.16$. Note that the general effect of the increase in bottom elevation is to draw the uniform-flow solutions together, causing the region of bounded solutions to shrink.

We now construct some sample solutions for the following topography:

$$\tilde{b} = \begin{cases} 0 & (|\xi - a| > l), \\ \tilde{b}_{max}[1 - (\xi - a)^2/l^2] & (|\xi - a| < l), \end{cases} \tag{5.8}$$

i.e. the bottom is flat except for a single, parabolic obstacle of height $b_{max}$ and half-width $l$. Away from the obstacle, the solution trajectories of figure 1 ($a$) are valid. The flow upstream of the obstacle can be one of three kinds.

*Case* 1: *uniform and slightly subcritical upstream flow*

Here the upstream condition is $g = f = 0$, and the flow remains uniform until the obstacle is encountered. As the fluid moves up the face of the obstacle, $\tilde{b}$ inceases and

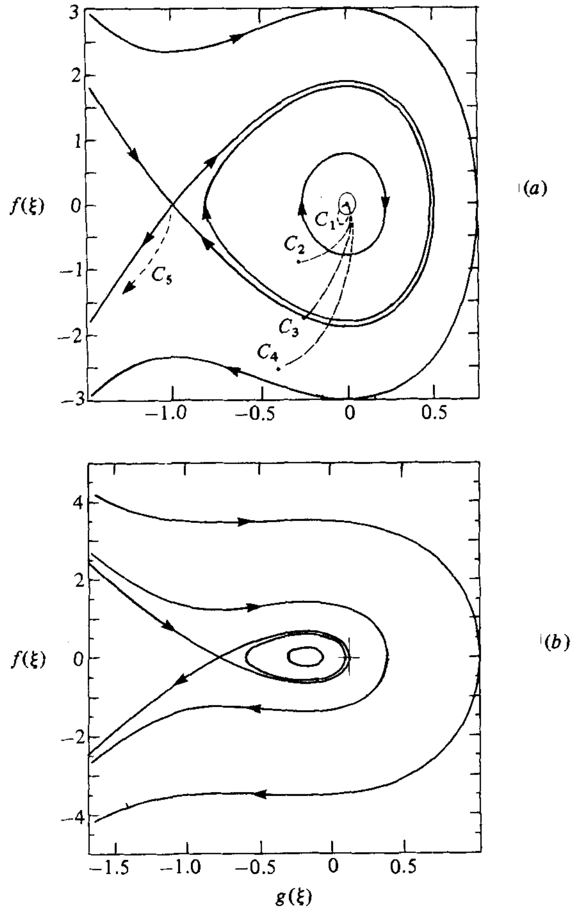


FIGURE 1. Phase planes for (5.1) and (5.2) with $Q^{(1)} = 1$, $\mathscr{B}^{(1)} = 0$ and constant bottom elevation $\tilde{b}$. The solid lines represent solution trajectories with arrows indicating the direction of increasing $\xi$. The dashed lines represent various possible excursions from shown trajectories when $\tilde{b}$ is allowed to vary with $\xi$ as described in the text. (a) $\tilde{b} = 0$; (b) $\tilde{b} = 0.16$.

the equilibrium (centre) point moves to the left, away from the origin, as in figure 1 (b). The initial tendency of the actual solution curve, however, is to remain at the origin until the closed, cnoidal wave trajectories begin to be swept by it. The solution curve then takes on a phase velocity tangent to the passing trajectories. After the crest of the obstacle is passed, $\tilde{b}$ decreases and the phase trajectories return to their original shapes. However, the solution curve may not return to the origin, and the downstream solution trajectory may be different from the upstream trajectory. Four different examples of the path that the solution curve might take over the obstacle are indicated by the dashed curves in figure 1 (a). The first (labelled $C_1$) places the downstream flow back at the origin, so that the upstream and downstream states are identical. The second (labelled $C_2$) connects the upstream solution to a cnoidal-wave trajectory, so that the obstacle produces lee waves. The third (labelled $C_3$) places the downstream flow on the solitary-wave trajectory, so that far downstream the flow becomes uniform and slightly supercritical. The final possibility (labelled $C_4$) places the solution onto an unbounded trajectory.

*Case 2: periodic upstream flow*

The upstream state could also consist of a uniform train of cnoidal waves. In this case the effect of the obstacle is similar to that described above. More specifically, three bounded downstream states are possible: a uniform, slightly subcritical flow; a periodic flow; and a partial solitary wave.

*Case 3: uniform supercritical flow upstream*

Suppose first that the upstream state lies at the saddle point $(0, -1)$. The flow is then slightly supercritical and uniform when the fluid first contacts the obstacle. As the bottom elevation increases, the saddle point is moved to the right but the solution curve is drawn down and to the left (curve $C_5$ in figure 1 (*a*)) and continues to move toward increasingly negative values of $g^{(1)}$ and $f^{(1)}$. After the obstacle is passed the solution curve is left on an unbounded trajectory.

To achieve a bounded solution, the upstream state must therefore lie not at the saddle point but elsewuere on the solitary-wave trajectory. In this case the upstream flow is uniform and slightly supercritical as $\xi \rightarrow -\infty$, but the crest of the solitary wave has started to form before the obstacle is encountered. As in the two previous cases, the downstream state can be one of three possible configurations, provided that the solution remains bounded.

A more thorough discussion of solutions with slightly supercritical upstream flow has been made by Kyner (1962), who derived an equation of the form (4.13) in connection with a potential flow. Here we concentrate on the situation in which the upstream flow is slightly subcritical, which would seem to be the more geophysically relevant case.

Figure 2 contains some sample solutions to (5.1) and (5.2) computed using a predictor–correlator method. The first three (figures 2 (*a*)–(*c*)) are computed for various values of $Q^{(1)}$ using $\mathscr{B}^{(1)} = 0$ and the topography (5.8). The upstream values $g = 0, f = 0$ are chosen so that the upstream flow is uniform and slightly subcritical. The solution of figure 2 (*a*) ($Q^{(1)} = 3.0$) corresponds roughly to the phase trajectory of $C_1$ of figure 1 (*a*), with the fluid dipping over the obstacle and returning to its upstream depth. That of figure 2 (*b*) ($Q^{(1)} = 0.86$) corresponds to the trajectory $C_2$, with cnoidal Kelvin waves appearing in the lee of the obstacle. Finally, the solution of figure 2 (*c*) ($Q^{(1)} = 0.8097$) corresponds to trajectory $C_3$ with a partial solitary wave asymptoting to a uniform, slightly supercritical flow in the lee of the obstacle.

The remainder of figure 2 contains solutions with other upstream states or obstacle shapes. Figure 2 (*d*) shows a flow that is periodic upstream of the obstacle and contains a partial solitary wave downstream. The flow in figure 2 (*e*) contains the same upstream state as that in figure 2 (*c*), but the obstacle has a cosine shape:

$$\tilde{b} = \begin{Bmatrix} 0 & (|\xi - a| > l), \\ \tfrac{1}{2}\tilde{b}_{\max}\{1 + \cos[\pi(\xi - a)/l]\} & (|\xi - a| \leqslant l). \end{Bmatrix} \tag{5.9}$$

The new shape causes lee waves to form, whereas the former shape leads to a partial solitary wave in the lee. Note the obstacle height $b_{\max}$ is the same in either case.

It has been assumed thus far that the coefficient $D_1 > 0$ in the transformation (4.15). If $D_1 < 0$, (5.1) and (5.2) become

$$\frac{\mathrm{d}f}{\mathrm{d}\xi} = (Q^{(1)} - g)g - \tilde{b} + \mathscr{B}^{(1)}, \quad \frac{\mathrm{d}g}{\mathrm{d}\xi} = f.$$

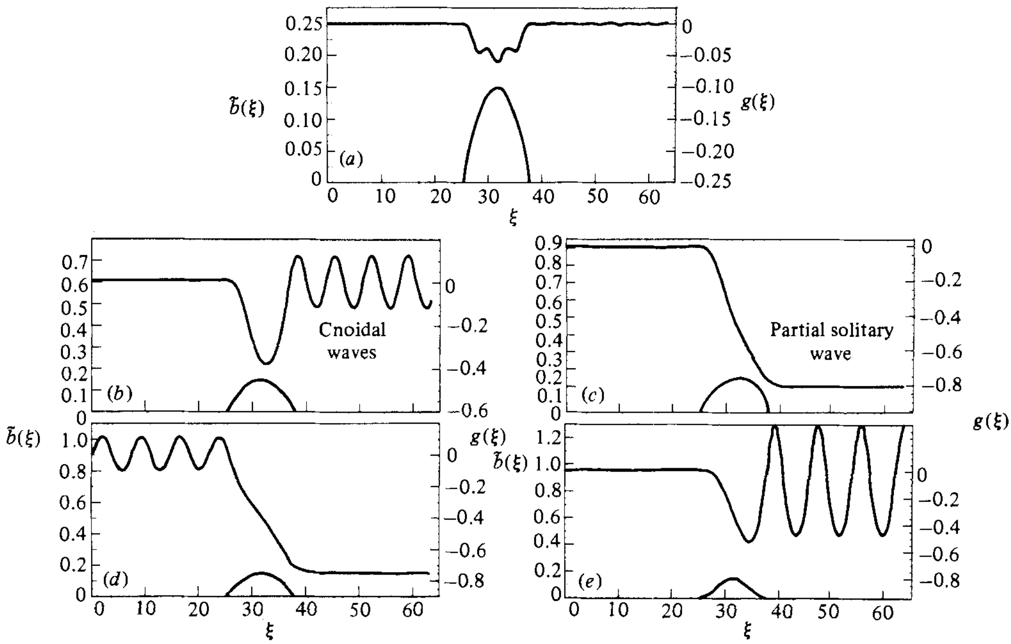


FIGURE 2. Numerical solutions to (5.1) and (5.2) for various upstream states and topography. In all cases the perturbation Bernoulli constant $\mathscr{B}^{(1)} = 0$, and the obstacles have height $\tilde{b}_{max} = 0.15$ and half-width $l = 6.5$. In (*a*)–(*d*) the obstacle shape is parabolic and is given by (5.8), while in (*e*) the shape is of a cosine and is given by (5.9). (*a*) $Q^{(1)} = 3.0$, $g = f = 0$ upstream; (*b*) $Q^{(1)} = 0.86$, $g = f = 0$ upstream; (*c*) $Q^{(1)} = 0.8097$, $g = f = 0$ upstream; (*d*) $Q^{(1)} = 0.7635$, $f = 0$ and $g = -0.3$ upstream at $x = 0$; (*e*) $Q^{(1)} = 0.8097, f = g = 0$ upstream. ($Q^{(1)}$ is proportional to the perturbation flow rate.)

The phase plane for the above equations can be obtained simply by displacing the phase plane for (5.1) and (5.2) an amount $Q^{(1)}$ in the positive $g$-direction. Furthermore, it is a simple matter to show that the right- and left-hand stationary points represent slightly subcritical and slightly supercritical flows in the displaced phase plane. The general along-channel behaviour of solutions for $D_1 < 0$ is thus identical with the case $D_1 > 0$.

If $D_1 = 0$ the dispersive term in (4.13) vanishes and solutions regain their long-wave behaviour in the $x$-direction. However, the author has been unable to contrive a basic flow for which $D_1 = 0$ other than the trivial case $\bar{h}^{(0)} = \phi^{-1}$ and $\hat{h}^{(0)} = 0$. The latter corresponds to a basic state of rest.

## 6. Upstream influence

The along-channel structure of several of the solutions to (5.1) and (5.2) is reminiscent of the solutions of classical hydraulics (Gill 1977). In figure 2(*a*), for example, the solution is nearly uniform and subcritical on either side of the obstacle, the only substantial departure from the uniform state occurring over the obstacle. This is also a property of the subcritical solution of hydraulics. The solution of figure 2(*c*) resembles the 'controlled' solution of hydraulics, in which a transition from subcritical to supercritical conditions occurs as fluid spills over the obstacle. However, the wavelike behaviour of the solutions in figures 2(*b*, *d*, *e*) is not found in classical hydraulics.

A well-known property of the 'controlled' solution of hydraulics is that the upstream state cannot be determined independently of the obstacle height. For example, a controlled, shallow, non-rotating free-surface flow with upstream energy per unit mass $E$ and flow rate $q$ is constrained by the relationship

$$b_{\max} = E - \tfrac{3}{2} g^{\frac{2}{3}} q^{\frac{2}{3}}, \tag{6.1}$$

where $b_{\max}$ is the height of the crest of the obstacle above the flat bottom upstream (Chow 1959). (The corresponding relationship for the rotating case is given by equations (4.2) and (4.3) in P1.) If (6.1) is initially satisfied and the obstacle height $b_{\max}$ is increased by a small amount to a new fixed value, $E$ and $q$ adjust to new values such that (6.1) is again satisfied. This adjustment is effected by a wave which is generated by the obstacle and moves upstream, bringing $E$ and $q$ back in line with (6.1). The new steady state is therefore controlled.

The time-dependent adjustment from one controlled state to another describes the process of *upstream influence*, first noted by Long (1954). Obstacles exercise upstream influence by sending signals upstream. These signals consist of gravity waves in the non-rotating case and Kelvin waves (see P1) when rotation exists. The communication is governed by a minimal principle which is implied by (6.1) (or the appropriate form thereof.) In the non-rotating case the minimal principle requires that the change of $E - \frac{3}{2} g^{\frac{2}{3}} q^{\frac{2}{3}}$ in response to an increase in $b_{\max}$ be the *smallest amount necessary for a steady well-behaved solution to be possible.* Now it is a fact that such solutions (controlled or otherwise) are not possible for $b_{\max} > E - \frac{3}{2} g^{\frac{2}{3}} q^{\frac{2}{3}}$, as the free-surface slope would become infinite in selected places (Gill 1977). Thus the minimum change in $E - \frac{3}{2} g^{\frac{2}{3}} q^{\frac{2}{3}}$ is the amount required to satisfy (6.1).

If $b_{\max} < E - \frac{3}{2} g^{\frac{2}{3}} q^{\frac{2}{3}}$ the steady solution is well-behaved but non-controlled. In this case, sufficiently small increases in $b_{\max}$ cause no change in status, and the minimum required change in $E - \frac{3}{2} g^{\frac{2}{3}} q^{\frac{2}{3}}$ is zero. The final upstream state is thus identical with the initial upstream state and no upstream influence occurs.

It may be asked whether upstream influence is also a property of solutions to (5.1) and (5.2). The question may be rephrased by asking whether an infinitesimal change in the heights (or shapes) or the obstacles in figure 2 will force a permanent change in the upstream conditions of any of the solutions. Suppose first that for all values $-\infty < \xi < \infty$ the solution curve lies within the region of phase space enclosed by the solitary-wave trajectory (see figure 1). Then an infinitesimal change in bottom topography will not affect the boundedness of the solution; such a solution may be found without altering the upstream flow. If, on the other hand, the solution trajectory lies along or touches the solitary-wave trajectory for any $\xi$, as in figures 2($c, d$), the boundedness of the solution is in jeopardy. Should the change in topography force the solution onto an outside trajectory, the downstream flow will become unbounded, and new upstream conditions ($Q^{(1)}$ and $\mathcal{B}^{(1)}$) are required. An increase in $(\frac{1}{2}Q^{(1)})^2 + \mathcal{B}^{(1)}$ would enlarge the region of bounded solutions by further separating the equilibrium points (see (5.4)). This influence over the upstream state is governed by the time-dependent non-linear dispersive equation in which (5.1) is embedded. If the process obeys the same minimal principles that govern non-dispersive adjustments, the solitary wave will be a common downstream state. This is an intuitive suggestion which is based on the fact that the solitary wave, of all bounded solutions, represents the 'outermost'. Therefore the minimal amount of change in $(\frac{1}{2}Q^{(1)})^2 + \mathcal{B}^{(1)}$ necessary to establish a bounded solution should be the amount needed to make the downstream solution curve coincide with the solitary wave trajectory.

It should be noted that the coincidence of the solution trajectory with the solitary-wave trajectory is not a *sufficient* condition for upstream influence. This can be seen through comparison of the solution of figure 2(*c*) with that of figure 2(*e*). Although the downstream part of the solution of figure 2(*c*) consists of a partial solitary wave, the change in obstacle shape necessitates no new upstream state. (Other obstacle shapes do, however, lead to an unbounded solution.)

Although the above conclusions are based on intuition rather than formal proof, there is some indirect experimental evidence in support. Pratt (1984) has carried out a series of experiments with non-rotating, nearly critical, free-surface flow over multiple obstacles; a system also governed by equations of the forms (5.1) and (5.2). For non-controlling obstacles, the downstream flow was found to consist of a train of cnoidal lee waves while the flow downstream of a controlling obstacle resembled a partial solitary wave.

## 7. Discussion

The presence of dispersive effects leads to several striking modifications of the rotating-hydraulics solutions described by Gill (1977) and in P1. The first is the possibility of oscillatory behaviour in the along-channel direction as a result of stationary cnoidal or solitary Kelvin waves. The second is the dependence of the upstream and/or downstream flow on obstacle shape as well as height (see figures 2(*c*, *e*)).

Several questions of great interest have arisen during the course of the analysis. First, how is upstream influence exercised by the obstacle? What is the messenger that conveys information regarding the obstacle shape and height to the upstream flow? Secondly, does a minimal principle govern the adjustment process by which the obstacle communicates with the upstream flow, as occurs in hydraulic theory? Finally, what determines whether the upstream flow (after adjustment) is uniform or oscillatory? Answers to these questions can be obtained by integrating the time-dependent equation in which (5.1) and (5.2) are imbedded, an exercise left for a future paper.

In the derivation of (5.1) and (5.2) it has been assumed that the thickness of the lower layer remains finite at all points. Gill (1977) has shown that this assumption can be violated when the channel width is greater than a certain critical value. This critical value is determined by (2.11). Under these conditions the active layer separates from the wall at $y = +w$ and the edge of the current forms a free streamline. Since the wall at $y = +w$ is no longer available to support Kelvin waves, it is unclear that equations of the form (5.1) and (5.2) continue to hold.

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## Appendix A

The uniform (in $\xi$) solutions described by (5.3) and (5.4) can be viewed as small corrections to the basic uniform flow. Consider the Froude number of the corrected flow, as obtained by replacing $\hat{h}^{(0)}$ and $\bar{h}^{(0)}$ by $\hat{h}^{(0)}+\delta^2\hat{h}^{(1)}$ and $\bar{h}^{(0)}+\delta^2\bar{h}^{(1)}$ in (2.12). If this substitution is carried out and the critical condition (3.7) applied, the result is

$$F_{\mathrm{d}} = 1+\delta^2\left(\frac{\hat{h}^{(1)}}{\hat{h}^{(0)}}-\frac{1}{2}\frac{\bar{h}^{(1)}}{\bar{h}^{(0)}}-\frac{T^4}{2}\frac{\bar{h}^{(0)}\bar{h}^{(1)}}{\hat{h}^{(0)2}}\right)+O(\delta^4).$$

If (3.8) is used to write $\bar{h}^{(1)}$ in terms of $\hat{h}^{(1)}$, the above expression can be written as

$$F_{\mathrm{d}} = 1+\tfrac{1}{2}\delta^2\alpha^3\bar{h}^{(0)}(3+\alpha^2T^4)\left[\hat{h}^{(1)}-\tfrac{1}{2}Q^{(1)}\alpha^{-1}(1+\alpha^2T^4)\,(3+\alpha^2T^4)^{-1}\right]+O(\delta^4), \quad (A\ 1)$$

where

$$\alpha = \bar{h}^{(0)}\hat{h}^{(0)^{-1}} < 0.$$

Suppose first that (5.6) holds, so that $g_{\mathrm{s}+} = g_{\mathrm{s}-} = g_{\mathrm{sc}}$. The corresponding value of $\hat{h}^{(1)}$, denoted by $\hat{h}^{(1)}_{\mathrm{sc}}$, can be computed from (4.15) as

$$\hat{h}^{(1)}_{\mathrm{sc}} = \tfrac{1}{2}Q^{(1)}\alpha^{-1}(1+\alpha^2T^4)\,(3+\alpha^2T^4)^{-1}.$$

The bracketed term in (A 1) thus vanishes and the uniform flow remains *critical*, $F_{\mathrm{d}} = 1$, through $O(\delta^2)$. If, on the other hand, (5.5) holds, two distinct values $g_{\mathrm{s}+}$ and $g_{\mathrm{s}-}$ exist such that

$$g_{\mathrm{s}+} > g_{\mathrm{sc}}, \quad g_{s-} < g_{\mathrm{sc}}.$$

From (4.15) it follows that the corresponding values of $\hat{h}^{(1)}$, denoted by $\hat{h}^{(1)}_{\mathrm{s}+}$ and $\hat{h}^{(1)}_{\mathrm{s}-}$, are characterized by

$$\hat{h}^{(1)}_{\mathrm{s}+} > \frac{\tfrac{1}{2}Q^{(1)}\alpha^{-1}(1+\alpha^2T^4)}{3+\alpha^2T^4}$$

and

$$\hat{h}^{(1)}_{\mathrm{s}-} < \frac{\tfrac{1}{2}Q^{(1)}\alpha^{-1}(1+\alpha^2T^4)}{3+\alpha^2T^4}.$$

From (A 1) it follows that $F_{\mathrm{d}} < 1$ in the first case, while $F_{\mathrm{d}} > 1$ in the second. In summary, the two uniform flows described by (5.3) and (5.4) are *slightly subcritical* and *slightly supercritical* when the elevation $\tilde{b}$ of the flat bottom is less than $(\tfrac{1}{2}Q^{(1)})^2+\mathscr{B}^{(1)}$. These two solutions arise as a bifurction from the uniform critical flow which occurs when $\tilde{b} = (\tfrac{1}{2}Q^{(1)})^2+\mathscr{B}^{(1)}$.

## Appendix B

An appropriate expression for the wavelengths of the cnoidal waves can be derived by rewriting (4.14) as

$$\frac{\mathrm{d}^2g}{\mathrm{d}\xi^2} = -(g-g_{\mathrm{s}+})\,(g-g_{\mathrm{s}-})$$

Attention is now restricted to solutions near the subcritical equilibrium point $g_{\mathrm{s}+}$ by writing $g = g_{\mathrm{s}+}+\tilde{g}$ and demanding that $|\tilde{g}| \ll g_{\mathrm{s}+}-g_{\mathrm{s}-} = 2[(\tfrac{1}{2}Q^{(1)})^2+\mathscr{B}^{(1)}-\tilde{b}]^{\frac{1}{2}}$. Equation (4.14) can now be written as

$$\frac{\mathrm{d}^2\tilde{g}}{\mathrm{d}\xi^2}+2[(\tfrac{1}{2}Q^{(1)})^2+\mathscr{B}^{(1)}-\tilde{b}]^{\frac{1}{2}}\tilde{g} = O(\tilde{g}^2).$$

Thus plane waves of small amplitude and of the form $\tilde{g} = \sin(2\pi\xi/\lambda)$ have wavelength

$$\lambda = 2^{\frac{1}{2}}\pi[(\tfrac{1}{2}Q^{(1)})^2 + \mathscr{B}^{(1)} - \tilde{b}]^{-\frac{1}{4}}.$$

This wavelength grows as the basic flow approaches criticality (i.e. as $\tilde{b} \rightarrow (\frac{1}{2}Q^{(1)})^2 + \mathscr{B}^{(1)}$). If the wave amplitude becomes finite, the wavelength will become amplitude-dependent.

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